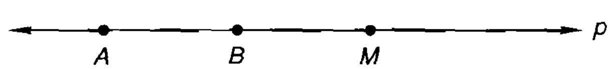
LESSON 1 Geometry Review

1.A points, lines, and rays

Some fundamental mathematical terms are impossible to define exactly. We call these terms **primitive terms** or **undefined terms.** We define these terms as best we can and then use them to define other terms. The words **point, curve, line,** and **plane** are primitive terms.

A point is a location. When we put a dot on a piece of paper to mark a location, the dot is not the point because a mathematical point has no size and the dot does have size. We say that the dot is the **graph** of the mathematical point and marks the location of the point. A curve is an unbroken connection of points. Since points have no size, they cannot really be connected. Thus, we prefer to say that a curve defines the path traveled by a moving point. We can use a pencil to graph a curve. These figures are curves.

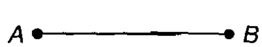
A mathematical line is a straight curve that has no ends. **Only one mathematical line can be drawn that passes through two designated points.** Since a line defines the path of a moving point that has no width, a line has no width. The pencil line that we draw marks the location of the mathematical line. When we use a pencil to draw the graph of a mathematical line, we often put arrowheads on the ends of the pencil line to emphasize that the mathematical line has no ends.



We can name a line by using a single letter (in this case, p) or by naming any two points on the line in any order. The line above can be called line AB, line BA, line AM, line MA, line BM, or line MB. Instead of writing the word *line*, a commonly used method is to write the letters for any two points on the line in any order and to use an overbar with two arrowheads to indicate that the line continues without end in both directions. All of the following notations name the line shown above. These notations are read as "line AB," "line BA," etc.

$$\overleftrightarrow{AB} \qquad \overleftrightarrow{BA} \qquad \overleftrightarrow{AM} \qquad \overleftrightarrow{MA} \qquad \overleftrightarrow{BM} \qquad \overleftrightarrow{MB}$$

We remember that a part of a line is called a **line segment** or just a **segment.** A line segment contains the endpoints and all points between the endpoints. A segment can be named by naming the two endpoints in any order. The following segment can be called segment AB or segment BA.



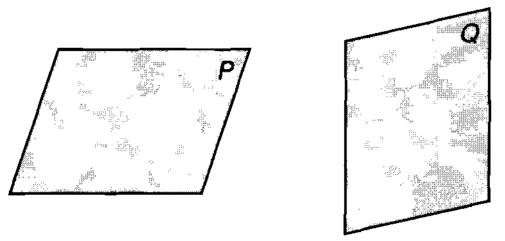
Instead of writing the word *segment*, we can use two letters in any order and an overbar with no arrowheads to name the line segment whose endpoints are the two given points. Therefore, $\overline{AB}$ means "segment AB" and $\overline{BA}$ means "segment BA." Thus, we can use either

$$\overline{AB} \quad \text{or} \quad \overline{BA}$$

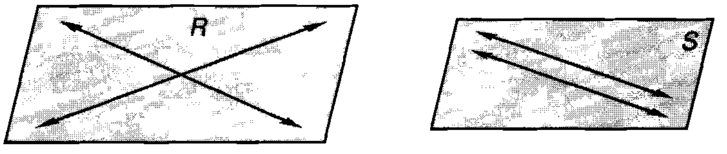
to name a line segment whose endpoints are A and B.

1.B planes

A mathematical line has no width and continues without end in both directions. A mathematical plane can be thought of as a flat surface like a tabletop that has no thickness and that continues without limit in the two dimensions that define the plane. Although a plane has no edges, we often picture a plane by using a four-sided figure. The figures below are typical of how we draw planes. We label and refer to them as plane P and plane Q, respectively.

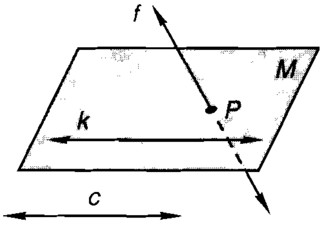


Just as two points determine a line, three noncollinear points determine a plane. As three noncollinear points also determine two intersecting straight lines, we can see that two lines that intersect at one point also determine a plane.



On the right, we see that two parallel lines also determine a plane. We say that lines that lie in the same plane are **coplanar.**

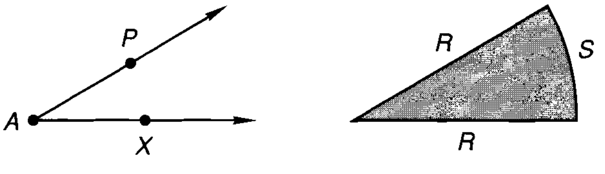
A line not in a plane is parallel to the plane if the line does not intersect the plane. If a line is not parallel to a plane, the line will intersect the plane and will do so at only one point. Here we show plane M and line k that lies in the plane. We also show line c that is parallel to the plane and line f that intersects the plane at point P.



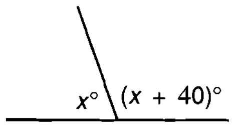
Skew lines are lines that are not in the same plane. Skew lines are never parallel, and they do not intersect. However, saying this is not necessary because if lines are parallel or intersect, they are in the same plane. Thus, lines k and f in the diagram above are skew lines because they are not both in plane M, and they do not form another plane because they are not parallel and they do not intersect.

1.C angles

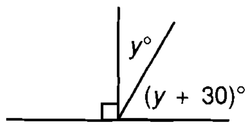
There is more than one way to define an angle. An angle can be defined to be the **geometric figure** formed by two rays that have a common endpoint. This definition says that the angle is the set of points that forms the rays, and that the measure of the angle is the measure of the opening between the rays. A second definition is that the angle is the **region** bounded by two radii and the arc of a circle. In this definition, the measure of the angle is the ratio of the length of the arc to the length of the radius.



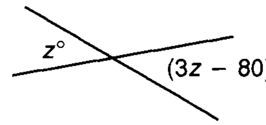
two angles is 90°, the angles are called **complementary angles.** On the left we show two adjacent angles whose measures sum to 180°. We can use this fact to write an equation to solve for x.



$$x + x + 40 = 180$$
$$2x = 140$$
$$x = 70$$



$$y + y + 30 = 90$$
$$2y = 60$$
$$y = 30$$

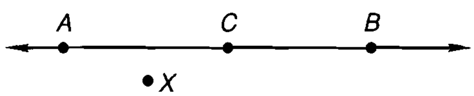


$$z = 3z - 80$$
$$-2z = -80$$
$$z = 40$$

In the center we show two adjacent angles whose measures sum to 90°. We can use this fact to write an equation and solve for y. Two pairs of **vertical angles** are formed by intersecting lines. Vertical angles have equal measures. On the right we show two vertical angles, write an equation, and solve for z.

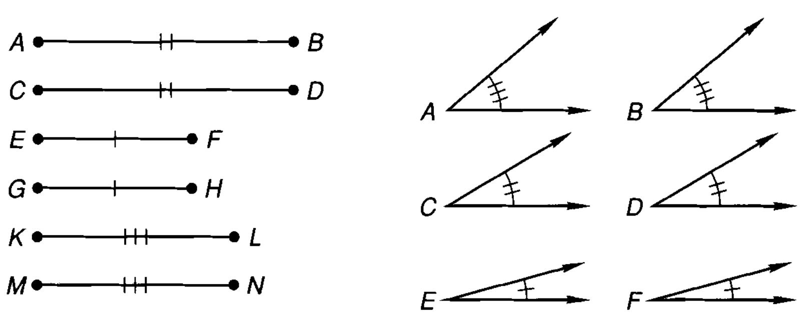
1.D betweenness, tick marks, and assumptions

One point is said to lie **between** two other points only if the points lie on the same line (are collinear). When three points are collinear, one and only one of the points is between the other two.



Thus, we can say that point C is between points A and B because point C belongs to the line segment determined by the two points A and B and is not an endpoint of this segment. Point X is not between points A and B because it is not on the same line (is not collinear) that contains points A and B.

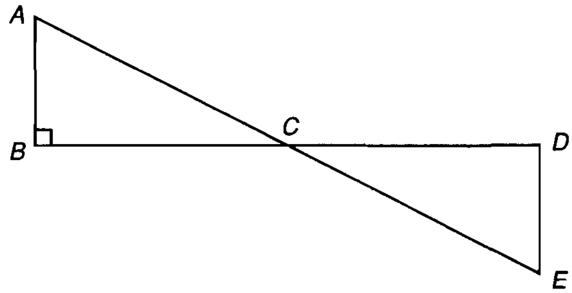
We will use **tick marks** on the figures to designate segments of equal length and angles of equal measure.



Here we have indicated the following equality of segment lengths: $AB = CD$, $EF = GH$, and $KL = MN$. The equality of angle measures indicated is $m\angle A = m\angle B$, $m\angle C = m\angle D$, and $m\angle E = m\angle F$.

In this book we will consider the formal proofs of geometry. We will use geometric figures in these proofs. When we do, some assumptions about the figures are permitted and others are not permitted. **It is permitted to assume that a line that appears to be a straight line is a straight line, but it is not permitted to assume that lines that appear to be perpendicular are perpendicular. Further, it is not permitted to assume that the measure of one angle or line segment is equal to, greater than, or less than the measure of another angle or line segment.** We list some permissible and impermissible assumptions on the following page.

PERMISSIBLE	NOT PERMISSIBLE
Straight lines are straight angles	Right angles
Collinearity of points	Equal lengths
Relative location of points	Equal angles
Betweenness of points	Relative size of segments and angles
	Perpendicular or parallel lines



In the figure above we may assume the following:

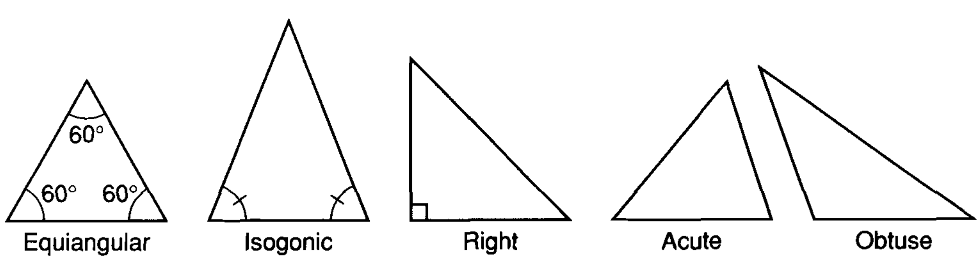
1. That the four line segments shown are segments of straight lines
2. That point C lies on line BD and on line AE
3. That point C lies between points B and D and points A and E

We may not assume:

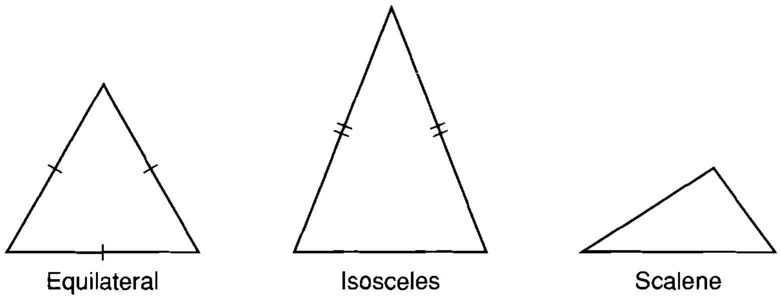
4. That $\overline{AB}$ and $\overline{DE}$ are of equal length
5. That $\overline{BC}$ and $\overline{CD}$ are of equal length
6. That $\angle D$ is a right angle
7. That $m\angle A$ equals $m\angle E$
8. That $\overline{AB}$ and $\overline{DE}$ are parallel

1.E triangles

Triangles have three sides and three angles. The sum of the measures of the angles in any triangle is 180°. Triangles can be classified according to the measures of their angles or according to the lengths of their sides. If the measures of all angles are equal, the triangle is **equiangular.** The Greek prefix *iso-* means "equal" and the Greek word *gonia* means "angle." We put them together to form *isogonic,* which means "equal angles." An **isogonic triangle** is a triangle in which the measures of at least two angles are equal. If one angle is a right angle, the triangle is a **right triangle.** If all angles have a measure less than 90°, the triangle is an **acute triangle.** If one angle has a measure greater than 90°, the triangle is an **obtuse triangle.** An **oblique triangle** is a triangle that is not a right triangle. Thus, acute triangles and obtuse triangles are also oblique triangles.



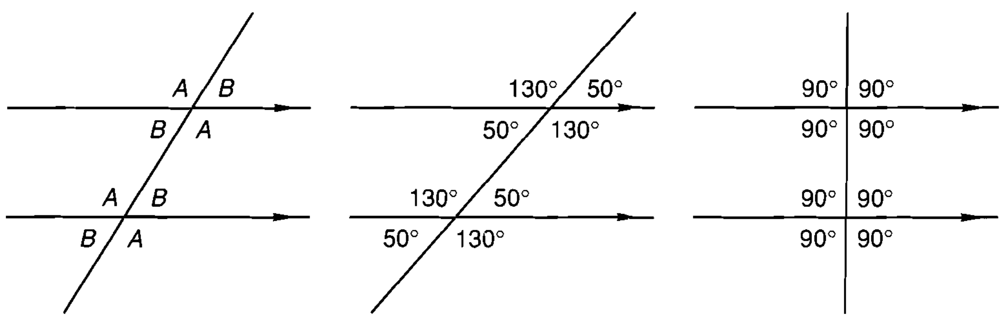
Triangles are also classified according to the relative lengths of their sides. The Latin prefix *equi-* means "equal" and the Latin word *latus* means "side." We put them together to form *equilateral,* which means "equal sides." An **equilateral triangle** is a triangle in which the lengths of all sides are equal. Since the Greek prefix *iso-* means "equal" and the Greek word *skelos* means "leg," we can put them together to form *isosceles,* which means "equal legs." An **isosceles triangle** is a triangle that has at least two sides of equal length. If all the sides of a triangle have different lengths, the triangle is called a **scalene triangle.**



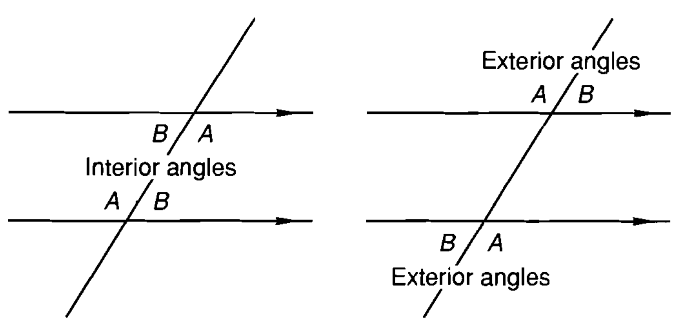
The lengths of sides of a triangle and the measures of the angles opposite these sides are related. If the lengths of the sides are equal, then the measures of angles opposite these sides are also equal. This means that every isogonic triangle is also an isosceles triangle and that every isosceles triangle is also an isogonic triangle. Every equilateral triangle is also an equiangular triangle and every equiangular triangle is also an equilateral triangle. If the measure of an angle in a triangle is greater than the measure of a second angle, the length of the side opposite the angle is greater than the length of the side opposite the second angle. The sum of the measures of the angles of any triangle is 180°.

1.F transversals; alternate and corresponding angles

We remember that two lines in a plane are called **parallel lines** if they never intersect. A **transversal** is a line that cuts or intersects one or more other lines in the same plane. **When two parallel lines are cut by a transversal, two groups of four angles whose measures are equal are formed. The four small (acute) angles have equal measures, and the four large (obtuse) angles have equal measures. If the transversal is perpendicular to the parallel lines, all eight angles are right angles.**

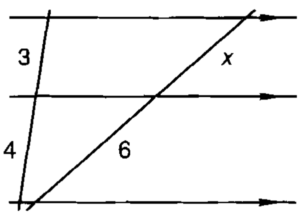


In the figures above, note the use of arrowheads to indicate that lines are parallel. In the left-hand figure, we have named the small angles *B* and the large angles *A*. In the center figure, we show a specific example where the small angles are 50° angles and the large angles are 130° angles. Also, note that **in each case, the sum of the measures of a small angle and a large angle is 180° because together the two angles always form a straight line.** The angles have special names that are useful. The four angles between the parallel lines are called **interior angles,** and the four angles outside the parallel lines are called **exterior angles.**



Some geometry textbooks will **postulate,** or in other words assume true without proof, one of the statements in each of the boxes and then prove the other statements as theorems. We decide to postulate all the statements in the boxes and use these statements to prove other geometric facts later.

When two transversals intersect three parallel lines, the parallel lines cut the transversals into line segments whose lengths are proportional, as we will show in Lesson 8. This fact will allow us to find the length of segment x in the diagram on the left. These segments are proportional. This means that the ratios of the lengths are equal.

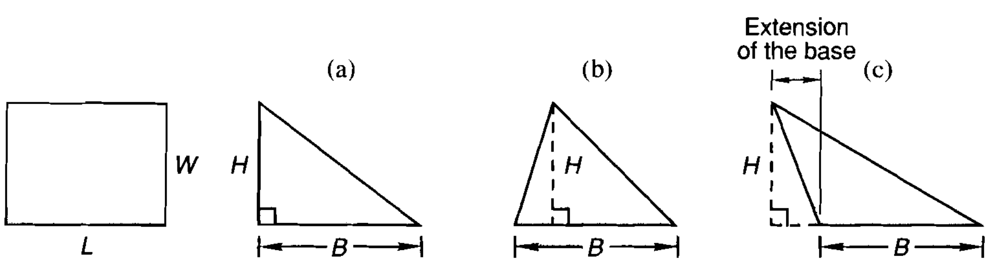


$$\frac{4}{6} = \frac{3}{x} \qquad \text{equal ratios}$$
$$4x = 18 \qquad \text{cross multiplied}$$
$$x = \frac{9}{2} \qquad \text{divided by 4}$$

We decided to put the lengths of the segments on the left on the top and the lengths of the segments on the right on the bottom.

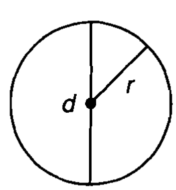
1.G area and sectors of circles

The area of a rectangle equals the product of the length and the width. The **altitude,** or **height,** of a triangle is the perpendicular distance from either the base of the triangle or an extension of the base to the opposite vertex. Any one of the three sides can be designated as the base. The altitude can (a) be one of the sides of the triangle, (b) fall inside the triangle, or (c) fall outside the triangle. When the altitude falls outside the triangle, we have to extend the base so that the altitude can be drawn. This extension of the base is not part of the length of the base. The area of any triangle equals one half the product of the base and the altitude.



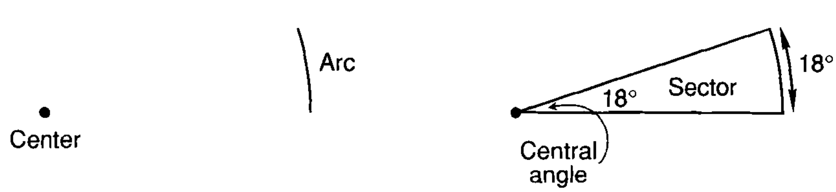
$A = L \times W$ $\qquad A = \frac{1}{2}BH$ $\qquad A = \frac{1}{2}BH$ $\qquad A = \frac{1}{2}BH$

The area of a circle equals πr^2, where r represents the **radius** of the circle. The perimeter of a circle is called the **circumference** of the circle and equals the product of π and the diameter of the circle. The length of the diameter equals twice the length of the radius.

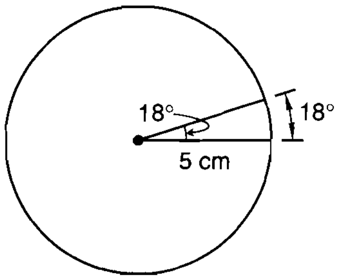


$$A = \pi r^2$$
$$C = \pi d = 2\pi r$$

An **arc** is two points on a circle and all the points on the circle between them. If we draw two radii to connect the endpoints of the arc to the center of the circle, the area enclosed is called a **sector** of the circle.



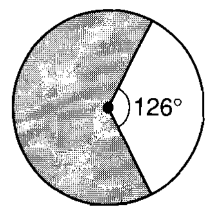
We define the degree measure of the arc to be the same as the degree measure of the **central angle** formed by the two radii. There are 360° in a circle. One degree of arc is $\frac{1}{360}$ of the circumference of the circle. Eighteen degrees of arc is $\frac{18}{360}$ of the circumference of the circle. The sector designated by 18° of arc is $\frac{18}{360}$ of the area of the circle. The length of an 18° arc is $\frac{18}{360}$ of the circumference of the circle. The radius of the circle shown below is 5 cm, so the area of the 18° sector is $\frac{18}{360}$ times the area of the whole circle. The length of an 18° arc is $\frac{18}{360}$ times the circumference of the whole circle.



$$\text{Area of } 18° \text{ sector} = \frac{18}{360} \times \pi(5 \text{ cm})^2 = 3.93 \text{ cm}^2$$

$$\text{Length of } 18° \text{ arc} = \frac{18}{360} \times 2\pi(5 \text{ cm}) = 1.57 \text{ cm}$$

example 1.1 The measure of the central angle of the unshaded sector is 126°. The radius of the circle is $2\sqrt{5}$ m. Find the area of the shaded region of the circle.



solution Since the full measure of a circle is 360°, the measure of the central angle of the shaded sector of the circle is 360° minus 126°, or 234°. The area of the shaded sector equals the product of the fraction of the central angle that is shaded and the area of the whole circle.

$$\text{Area of } 234° \text{ sector} = \frac{234}{360} \times \pi(2\sqrt{5} \text{ m})^2 = \mathbf{13\pi \text{ m}^2}$$

1.H concept review problems

Virtually all the problems in the problem sets are designed to afford the student practice with concepts or skills. Often, a problem may be contrived so that it requires the use of a particular technique or the application of a particular concept. However, there are some problems that defy simple classification. These may be problems that appear very difficult but can be easily solved with clever reasoning or a "trick." These may be problems that can be solved using either very long and tedious calculations or may be very easily solved through some "shortcut" requiring a deep understanding of the underlying concepts. Knowing what "shortcuts" and "tricks" to use requires experience. We do not believe one can become a good problem solver by reading about the philosophy of problem solving. One learns the art of problem solving by solving problems. What we will do is provide conceptually oriented review problems at the end of many problem sets to permit exposure to a wide spectrum of problems, many of which appear in some similar form on standardized exams such as the ACT and SAT. Through time and practice, students gain confidence, experience, and competence solving these types of problems.

Problems that compare the values of quantities come in many forms and can be used to provide practice in mathematical reasoning. In these problems, a statement will be made about two quantities A and B. **The correct answer is A if quantity A is greater and is B if quantity B is greater. The correct answer is C if the quantities are equal and is D if insufficient information is provided to determine which quantity is greater.**

example 1.2 Let x and y be real numbers. If $x > 0$ and $y < 0$, compare: A. $x + y$ B. $x - y$

problem set 1

1. Find the measure of angle A if five times the complement of angle A is 150° greater than the supplement of angle A. Remember that the complement of angle A has a measure of $90° - A$ degrees and the supplement of angle A has a measure of $180° - A$ degrees.

2. Find the measure of angle B if seven times the complement of angle B exceeds twice the supplement of angle B by 220°.

3. Find an angle such that 4 times its complement equals 200°.

4. Twenty percent of the molybdenum fused. If 1420 grams did not fuse, what was the total mass of molybdenum used?

5. The ratio of pusillanimous brave men to oxymorons on the battlefield is 17 to 2. If the total of both on the battlefield is 342, how many are oxymorons?

Simplify:

6. $\dfrac{2^{-3}x^0(x^2)}{x^{-3}xy^{-3}y}$

7. $-(-3 - 2) + 4(-2) + \dfrac{1}{-2^{-3}} - (-2)^{-3}$

8. Simplify by adding like terms: $\dfrac{xy}{y^{-2}} - \dfrac{3x^4y^4}{x^3y} + \dfrac{7xy^{-2}}{xy^{-3}}$

9. Expand: $\dfrac{x^0y^{-2}x}{x^3y}\left(\dfrac{x^2y}{m} - \dfrac{3x^4y^2}{m^{-2}}\right)$

10. Solve: $3^0(2x - 5) + (-x - 5) = -3(x^0 - 2)$

11. Evaluate: $xy - x(x - y^0)$ if $x = 2$ and $y = -\dfrac{1}{2}$

12. Add: $\dfrac{2}{x} + \dfrac{x}{x + 1}$

13. Multiply: $(2x + 3)(2x^2 + 2x + 2)$

14. Use elimination to solve: $\begin{cases} N_N + N_D = 50 \\ 5N_N + 10N_D = 450 \end{cases}$

15. Use substitution to solve: $\begin{cases} N_P + N_D = 50 \\ N_P + 10N_D = 140 \end{cases}$

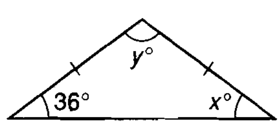
16. Use N_N as the variable for the number of nickels and N_D as the variable for the number of dimes to write two equations that correspond to the following statement and solve for N_N and N_D:

    The number of nickels exceeds the number of dimes by 7. The total value of the coins is \$1.55.

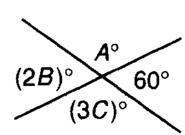
17. What do we mean when we say that two line segments are congruent?

18. What is an acute angle?

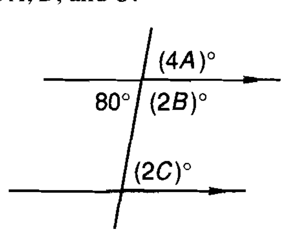
19. Find x and y.



20. Find A, B, and C.



21. Find A, B, and C.



22. Find x.