REVIEW LESSON A The real numbers · Fundamental concept review

A.A The real numbers

The numbers that we naturally use to count make up the set called the ***natural numbers,*** or the ***counting numbers,*** or the ***positive integers.*** We use the symbol $\mathbb{N}$ to represent this set.

$$\mathbb{N} = \{1, 2, 3, \cdots\}$$

The negatives of these numbers are called the ***negative integers.*** If we include the number zero with the positive and the negative integers, we can designate the set of integers. The symbol $\mathbb{Z}$ is often used to represent this set. This symbol comes from the first letter in the German word *zahlen*, which means "integer."

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$$

Any number that can be written as a quotient (fraction) of integers (division by zero excluded) is called a ***rational number.*** We use the symbol $\mathbb{Q}$ for quotient to designate this set. The following numbers are rational numbers.

$$0 \qquad 4 \qquad 35 \qquad \frac{-7}{23} \qquad \frac{45}{14} \qquad \frac{43}{6} \qquad \frac{19}{73}$$

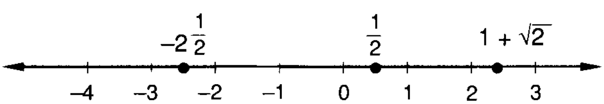
Any number that cannot be written as a quotient of integers is called an ***irrational number.*** We do not have a symbol for this set. Examples of irrational numbers are

$$\sqrt{2} \qquad \pi \qquad e \qquad \sqrt[3]{13} \qquad \sqrt[5]{41}$$

The set of ***real numbers*** includes all members of the set of rational numbers and all members of the set of irrational numbers. We use the symbol $\mathbb{R}$ to represent the set of real numbers. Every natural number is an integer. Every integer is a rational number, and every rational number is a real number. If we use $\subset$ to mean "a subset of," we can write

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

The real numbers make up an ordered set, for the members of the set of real numbers can be arranged in order, which we indicate when we draw a real number line.



Each point on the number line is associated with a unique number called the ***coordinate*** of the point. When we graph a number, we place a dot on the number line to indicate the position of the point that has this number as its coordinate. On the number line above we have graphed $\frac{1}{2}$, $1 + \sqrt{2}$, and $-2\frac{1}{2}$.

The order properties of real numbers are listed in the following box.

> ORDER PROPERTIES
>
> If x, y, and z represent real numbers, then
>
> 1. **Trichotomy.** Exactly one of the following is true:
>
> $$x < y \qquad \text{or} \qquad x = y \qquad \text{or} \qquad x > y$$
>
> 2. **Transitivity.** If $x < y$ and $y < z$, then $x < z$.
> 3. **Addition.** If $x < y$, then $x + z < y + z$.
> 4. **Multiplication.** If z is positive and $x < y$, then $xz < yz$. If z is negative and $x < y$, then $xz > yz$.

The set of real numbers is closed under the operations of addition and multiplication, since the sum of any two real numbers is a real number and the product of any two real numbers is a real number. The real numbers constitute a ***field.*** The properties of a field are shown in the following box.

> THE FIELD PROPERTIES
>
> If x, y, and z represent real numbers, then
>
> 1. **Commutative laws.** $x + y = y + x$ and $xy = yx$.
> 2. **Associative laws.** $x + (y + z) = (x + y) + z$ and $x(yz) = (xy)z$.
> 3. **Distributive law.** $x(y + z) = xy + xz$.
> 4. **Identity elements.** There are two distinct numbers 0 and 1 satisfying $x + 0 = x$ and $x \cdot 1 = x$.
> 5. **Inverses.** Each number x has an additive inverse (also called a ***negative***), $-x$, satisfying $x + (-x) = 0$. Also, each number x except 0 has a multiplicative inverse (also called a ***reciprocal***), x^{-1}, satisfying $x \cdot x^{-1} = 1$.

A.B Fundamental concept review

Now we will review some of the fundamental concepts from algebra whose use is required in the calculus problems in this book. Rather than use an expository review, we will review by working problems whose solutions require the applications of the concepts. We assume in each step that no denominator equals zero.

Example A.1 Solve $y = v\left(\frac{a}{x} + \frac{b}{mc}\right)$ for c.

Solution We will (1) eliminate parentheses, (2) multiply by the least common multiple of the denominators and simplify, (3) put all terms containing c on one side of the equals sign, and (4) factor c and then divide.

$y = \dfrac{va}{x} + \dfrac{vb}{mc}$	eliminated parentheses
$xmc \cdot y = xmc \cdot \dfrac{va}{x} + xmc \cdot \dfrac{vb}{mc}$	multiplied by LCM of denominators
$xmcy = mcva + xvb$	simplified

$$xmcy - mcva = xvb \qquad \text{rearranged}$$

$$c(xmy - mva) = xvb \qquad \text{factored}$$

$$\mathbf{c = \frac{xvb}{xmy - mva}} \qquad \text{divided}$$

Example A.2 Simplify: (*a*) $\dfrac{x}{a + \dfrac{m}{1 + \frac{c}{d}}}$ (*b*) $\dfrac{\dfrac{a}{x^2} + \dfrac{b}{x}}{\dfrac{m}{x^2} + \dfrac{k}{xc}}$

Solution (*a*) When there is no equals sign, the denominators cannot be eliminated, but we can write this expression as a simple fraction. We (1) add, (2) simplify, (3) add, and (4) simplify.

(1) $$\frac{x}{a + \dfrac{m}{\dfrac{d + c}{d}}} \qquad \text{added}$$

(2) $$\longrightarrow \frac{x}{a + \dfrac{md}{d + c}} \qquad \text{simplified}$$

(3) $$\longrightarrow \frac{x}{\dfrac{a(d + c) + md}{d + c}} \qquad \text{added}$$

(4) $$\longrightarrow \mathbf{\frac{x(d + c)}{a(d + c) + md}} \qquad \text{simplified}$$

(*b*) There is no equals sign in this expression, so the denominators cannot be eliminated. We (1) add above and below and (2) simplify.

(1) $$\frac{\dfrac{a + bx}{x^2}}{\dfrac{mc + kx}{x^2c}} \qquad \text{added above and below}$$

(2) $$\longrightarrow \mathbf{\frac{c(a + bx)}{mc + kx}} \qquad \text{simplified}$$

Example A.3 Simplify: $\dfrac{4 + \sqrt{2}}{3 - 2\sqrt{2}}$

Solution We multiply above and below by $3 + 2\sqrt{2}$ and simplify.

$$\frac{4 + \sqrt{2}}{3 - 2\sqrt{2}} \cdot \frac{3 + 2\sqrt{2}}{3 + 2\sqrt{2}} \longrightarrow \frac{16 + 11\sqrt{2}}{9 - 8} = \mathbf{16 + 11\sqrt{2}}$$

Example A.4 Simplify: $3\sqrt{\frac{3}{2}} - 4\sqrt{\frac{2}{3}} - \sqrt{24}$

Solution First we use two steps to rationalize the denominator.

$$3\frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} - 4\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} - 2\sqrt{6} \longrightarrow \frac{3\sqrt{6}}{2} - \frac{4\sqrt{6}}{3} - 2\sqrt{6}$$

We finish by adding these three terms, using 6 as a common denominator.

$$\frac{9\sqrt{6}}{6} - \frac{8\sqrt{6}}{6} - \frac{12\sqrt{6}}{6} = \mathbf{\frac{-11\sqrt{6}}{6}}$$

Example A.5 Simplify: $2\sqrt{-2}\sqrt{2} + 3i\sqrt{2} - \sqrt{-2}\sqrt{-2}$

Solution We will use three steps to simplify.

$$2\sqrt{2}i\sqrt{2} + 3i\sqrt{2} - \sqrt{2}i\sqrt{2}i \longrightarrow 4i + 3\sqrt{2}i + 2 \longrightarrow \mathbf{2 + (4 + 3\sqrt{2})i}$$

Example A.6 Simplify: $\dfrac{2i^2 - 3i + 4}{i^2 + 2i - 1}$

Solution First we simplify above and below. Then we multiply above and below by the conjugate of the denominator.

$$\frac{2 - 3i}{-2 + 2i} = \frac{2 - 3i}{-2 + 2i} \cdot \frac{-2 - 2i}{-2 - 2i} = \frac{-10 + 2i}{8} = \mathbf{\frac{-5}{4} + \frac{1}{4}i}$$

Example A.7 Simplify: (*a*) $\dfrac{y^{x+3}y^{x/2-1}z^{a}}{y^{(x-a)/2}z^{(x-a)/3}}$ (*b*) $x^{3/4}\sqrt{xy}\,x^{1/2}\sqrt[3]{x^4}$

Solution (*a*) First we rearrange and then we add exponents of like bases.

$$y^{x+3+x/2-1-x/2+a/2}z^{a-x/3+a/3} = \mathbf{y^{x+2+a/2}z^{4a/3-x/3}}$$

(*b*) Next we replace the radicals with fractional exponents and then add the exponents of like bases.

$$x^{3/4}x^{1/2}y^{1/2}x^{1/2}x^{4/3} \longrightarrow \mathbf{x^{37/12}y^{1/2}}$$

Example A.8 Factor: $4a^{3m+2} - 16a^{3m}$

Solution If each term is written in factored form, the common factor $4a^{3m}$ can be determined by inspection. Then we factor out the common factor and finish by factoring $a^2 - 4$.

$$(4)a^{3m}a^2 - (4)(4)a^{3m} = 4a^{3m}(a^2 - 4) \qquad \text{common factor}$$
$$= \mathbf{4a^{3m}(a + 2)(a - 2)} \qquad \text{factored } a^2 - 4$$

Example A.9 Factor: (*a*) $8a^3 - b^3c^6$ (*b*) $m^3 + x^3y^6$

Solution (*a*) We know that the difference of two cubes $F^3 - S^3$ can be factored as $(F - S)(F^2 + FS + S^2)$, where F is the first term and S is the second term. We note that expression (*a*) can be written as the difference of the two cubes. Then the factored form can be written by using the factored form of $F^3 - S^3$ as a guide.

$$(2a)^3 - (bc^2)^3 = \mathbf{(2a - bc^2)(4a^2 + 2abc^2 + b^2c^4)}$$

(*b*) The sum of two cubes $(F^3 + S^3)$ has $(F + S)$ as one factor. The other factor has F^2 as the first term and S^2 as the third term. The middle term is $-FS$.

$$F^3 + S^3 = (F + S)(F^2 - FS + S^2)$$

The expression $m^3 + x^3y^6$ can be written as the sum of two cubes. We can then write the factored form by inspection by comparing it to the factored form of $F^3 + S^3$.

$$(m)^3 + (xy^2)^3 = \mathbf{(m + xy^2)(m^2 - mxy^2 + x^2y^4)}$$

Example A.10 Evaluate: (a) $\frac{14!}{6!11!}$ (b) $\sum_{j=0}^{3} \frac{2^j}{j+1}$ (c) $\sum_{i=1}^{4} 3$

Solution

(a) $\frac{\overset{7}{\cancel{14}} \cdot 13 \cdot \cancel{12} \cdot \cancel{11!}}{6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{11!}} = \frac{7 \cdot 13}{30} = \mathbf{\frac{91}{30}}$

(b) $\frac{2^0}{0+1} + \frac{2^1}{1+1} + \frac{2^2}{2+1} + \frac{2^3}{3+1} = 1 + 1 + \frac{4}{3} + 2 = \mathbf{\frac{16}{3}}$

(c) $3 + 3 + 3 + 3 = \mathbf{12}$

Problem set A

Problem sets in this book will end with two or three concept review questions.

Problems that compare the values of quantities come in many forms and can be used to provide practice in mathematical reasoning. In these problems, a statement will be made about two quantities A and B. The correct answer is A if quantity A is greater and is B if quantity B is greater. The correct answer is C if the quantities are equal and is D if insufficient information is provided to determine which quantity is greater.

1. Compare: *A.* $7\frac{1}{5}$ ft^2 *B.* 0.8 yd^2

2. If $x = t$, compare: *A.* $7(2t - 2x)$ *B.* $-6(3t - 3x)$

3. If $4 < x < 9$ and $2 < y < 14$, compare: *A.* x *B.* y

4. If a is the average of 3 and 6, compare: *A.* $3a$ *B.* $a + 6$

5. Solve for R_1: $\frac{m}{x} = y\left(\frac{1}{R_1} + \frac{a}{R_2}\right)$

Simplify:

6. $a + \cfrac{1}{a + \cfrac{1}{a}}$

7. $\cfrac{1}{a + \cfrac{1}{x + \cfrac{1}{m}}}$

8. $\frac{x^2y}{1 + m^2} + \frac{x}{y}$

9. $\frac{4 - 3\sqrt{2}}{8 - \sqrt{2}}$

10. $3\sqrt{-4} + 2\sqrt{4} - \sqrt{-9}$

11. $-i^2 - 4i^3 + 2\sqrt{-2}\sqrt{-2}$

12. $\frac{3 + 2i}{4 - i}$

13. $\frac{x^a y^{a+b}}{x^{-a/2} y^{b-1}}$

14. $\frac{m^{x+2} b^{x-2}}{m^{2x/3} b^{-3x/2}}$

15. $\sqrt{xy}\, x^{2/3} y^{-3/2}$

16. Solve: $\begin{cases} 2x + 3y = -4 \\ x - 2z = -3 \\ 2y - z = -6 \end{cases}$

Factor:

17. $16a^{4m+3} - 8a^{2m+3}$

18. $a^2b^{2x+2} - ab^{2x+1}$

19. $a^6 - 27b^3c^3$

20. $x^3y^6 + 8m^{12}$